

# ON THE MULTIPLICITIES OF THE IRREDUCIBLE HIGHEST WEIGHT MODULES OVER KAC-MOODY ALGEBRAS

SERGEY MOZGOVOY

**ABSTRACT.** We prove that the weight multiplicities of the integrable irreducible highest weight module over the Kac-Moody algebra associated to a quiver are equal to the root multiplicities of the Kac-Moody algebra associated to some enlarged quiver. To do this, we use the Kac conjecture for indivisible roots and a relation between the Poincaré polynomials of quiver varieties and the Kac polynomials, counting the number of absolutely irreducible representations of the quiver over finite fields. As a corollary of this relation, we get an explicit formula for the Poincaré polynomials of quiver varieties, which is equivalent to the formula of Hausel [7].

## 1. INTRODUCTION

Let  $(\Gamma, I)$  be a finite quiver without loops, where  $I$  is the set of vertices. The underlying graph of  $\Gamma$  defines a symmetric generalized Cartan matrix  $C$  with  $c_{ij}$  being equal to minus the number of edges connecting the vertices  $i$  and  $j$  if  $i \neq j$  and  $c_{ii} = 2$ . Let  $\mathfrak{g}(\Gamma)$  be the corresponding Kac-Moody algebra [10]. We will identify  $\mathbb{Z}^I$  with its root lattice. Define the quadratic form  $T$  on  $\mathbb{Z}^I$ , called the Tits form, by the matrix  $\frac{1}{2}C$ . For any  $\nu = (\nu_i)_{i \in I} \in \mathbb{N}^I$ , we define  $\bar{\nu} := \sum_i \nu_i \omega_i$ , where  $(\omega_i)_{i \in I}$  are the fundamental weights (which are in general not unique). Our purpose is to relate the multiplicities of the integrable irreducible highest weight module  $L(\bar{\nu})$  with the root multiplicities of the Kac-Moody algebra associated to some enlarged quiver.

We define this enlarged quiver  $\Gamma_*$  by adjoining to  $\Gamma$  a new vertex  $*$  and  $\nu_i$  arrows from  $*$  to  $i$ , for each  $i \in I$ . Its set of vertices is  $I_* = I \cup \{*\}$ . As before, we may consider the Kac-Moody algebra  $\mathfrak{g}(\Gamma_*)$  associated to the quiver  $\Gamma_*$ . For any  $\alpha \in \mathbb{Z}^{I_*}$  and any  $k \in \mathbb{Z}$ , we consider the pair  $(\alpha, k)$  as an element of  $\mathbb{Z}^{I_*}$ . Our main result is the following

**Theorem 1.** *For any  $\alpha \in \mathbb{N}^I$ , we have  $\dim L(\bar{\nu})_{\bar{\nu}-\alpha} = \dim \mathfrak{g}(\Gamma_*)_{(\alpha,1)}$ .*

For example, it allows to perform an explicit computation of the multiplicities of  $L(\bar{\nu})$  using the Peterson recursive formula for the root multiplicities, see [17, 14]. It also allows to write down the character formula for the part  $\oplus_{\alpha} \mathfrak{g}(\Gamma_*)_{(\alpha,1)}$  of the Kac-Moody algebra  $\mathfrak{g}(\Gamma_*)$  and one can conjecture that similar formulas for the other parts of  $\mathfrak{g}(\Gamma_*)$  exist as well.

To prove the theorem, we use quiver varieties. Let  $T_*$  be the Tits form of the quiver  $\Gamma_*$  and let  $d = d(\alpha, \nu) := 1 - T_*(\alpha, 1) = \alpha \cdot \nu - T(\alpha)$ , where  $\alpha \cdot \nu = \sum_i \alpha_i \nu_i$ . It is known that  $\dim L(\bar{\nu})_{\bar{\nu}-\alpha}$  equals  $h_c^{2d}(\mathcal{M})$ , where  $\mathcal{M} = \mathcal{M}(\alpha, \nu)$  is a quiver variety having dimension  $2d$ , see [16, 18]. Using the trick of Crawley-Boevey [3] one can

consider the quiver variety  $\mathcal{M}(\alpha, \nu)$  as a certain moduli space of representations of the double quiver  $\bar{\Gamma}_*$  of the quiver  $\Gamma_*$  (obtained by adjoining reverse arrows for all arrows in  $\Gamma_*$ ). This allows us to relate the Poincaré polynomial of  $\mathcal{M}$  (and thus  $h_c^{2d}(\mathcal{M})$ ) to certain invariants of the quiver  $\Gamma_*$ .

Namely, for any quiver  $(\Gamma, I)$  and any  $\alpha \in \mathbb{N}^I$ , let  $a_\alpha(\Gamma, q)$  be the number of absolutely indecomposable representations of  $\Gamma$  over  $\mathbb{F}_q$  of dimension  $\alpha$ . It is proved in [9] that  $a_\alpha(\Gamma, q)$  are polynomials in  $q$  with integer coefficients. Moreover,  $a_\alpha(\Gamma, q) \neq 0$  if and only if  $\alpha$  is a root of  $\mathfrak{g}(\Gamma)$  and  $a_\alpha(\Gamma, q) = 1$  if and only if  $\alpha$  is a real root. We call these polynomials the Kac polynomials of the quiver  $\Gamma$ . It was conjectured by Kac [9] that  $a_\alpha(\Gamma, 0) = \dim \mathfrak{g}(\Gamma)_\alpha$ . This conjecture was proved for indivisible roots in [4]. The proof of the full conjecture was announced by Hausel [7]. We derive Theorem 1 from the Kac conjecture for indivisible roots and our second result, which is

**Theorem 2.** *We have  $\sum_i h_c^{2i}(\mathcal{M})q^i = q^d a_{(\alpha, 1)}(\Gamma_*, q)$  and  $h_c^{2i+1}(\mathcal{M}) = 0$  for  $i \geq 0$ .*

The proof of the theorem is based on the paper [4]. In the course of the proof we additionally show that the polynomial  $p(\mathcal{M}, q) := \sum_i h_c^{2i}(\mathcal{M})q^i$  counts the number of points in  $\mathcal{M}(\mathbb{F}_q)$  for a finite field  $\mathbb{F}_q$  of a sufficiently large characteristic.

Notice, that the Poincaré polynomial (of cohomologies with compact support) of  $\mathcal{M}$  equals  $p(\mathcal{M}, t^2)$ . Theorem 2 allows to calculate the Poincaré polynomial of  $\mathcal{M}$  using a formula for  $a_\alpha(\Gamma, q)$  for arbitrary quiver  $(\Gamma, I)$ , see [8, Theorem 4.6] or [14, Theorem 2]. Following [20], we define the function  $r_\alpha(\Gamma, q)$  by the formula

$$\frac{1}{\#\mathrm{GL}_\alpha(\mathbb{F}_q)} \#\{(g, x) \in (\mathrm{GL}_\alpha \times \mathcal{R}(\Gamma, \alpha))(\mathbb{F}_q) \mid gx = x, g \text{ is unipotent}\},$$

where  $\mathrm{GL}_\alpha$  and  $\mathcal{R}(\Gamma, \alpha)$  are defined in Section 2. There is an explicit formula for  $r_\alpha(\Gamma, q)$  and an easy relation between the generating functions

$$a(\Gamma, q) := \sum_{\alpha \in \mathbb{N}^I} a_\alpha(\Gamma, q)x^\alpha \quad \text{and} \quad r(\Gamma, q) := \sum_{\alpha \in \mathbb{N}^I} r_\alpha(\Gamma, q)x^\alpha,$$

see Section 5. Using the relation between  $a(\Gamma, q)$  and  $r(\Gamma, q)$  we obtain from Theorem 2

**Theorem 3.** *We have*

$$\sum_{\alpha} q^{-d(\alpha, \nu)} p(\mathcal{M}(\alpha, \nu), q)x^\alpha = (q-1) \frac{\sum_{\alpha} r_{(\alpha, 1)}(\Gamma_*, q)x^\alpha}{r(\Gamma, q)}.$$

This formula is equivalent to the formula of Hausel [7, Theorem 5]. As well as Theorem 2 it gives an effective way to calculate the Poincaré polynomials of quiver varieties.

In the second section we recall the definition of the moduli spaces of quiver representations according to [11]. In the third section we recall the basic properties of quiver varieties. Section 4 is devoted to the proof of Theorem 1 and Theorem 2. In Section 5 we recall the relation between Kac polynomials and function  $r(\Gamma, q)$  and then prove Theorem 3.

## 2. MODULI SPACES OF QUIVER REPRESENTATIONS

In this section we follow closely [4]. Let  $\Gamma$  be a quiver and  $I$  be its set of vertices. For any arrow  $h \in \Gamma$ , we denote by  $h'$  and  $h''$  its source and target respectively. We denote by  $\bar{\Gamma}$  the double of  $\Gamma$ , obtained from it by adjoining reverse arrows for

all arrows in  $\Gamma$ . For any  $h \in \bar{\Gamma}$ , we denote by  $\bar{h}$  the opposite arrow from  $\bar{\Gamma}$ . For any  $\alpha, \nu \in \mathbb{Z}^I$  we define  $\alpha \cdot \nu := \sum_{i \in I} \alpha_i \nu_i$ .

Let  $R$  be a commutative ring. For any  $I$ -graded free  $R$ -module  $V$  of finite rank, we define  $\dim V := (\text{rk } V_i)_{i \in I} \in \mathbb{N}^I$ . Given two  $I$ -graded free  $R$ -modules  $V$  and  $W$ , we denote the module of  $I$ -graded morphisms between them by  $\text{Hom}_I(V, W)$ .

Let  $\alpha \in \mathbb{N}^I$  and let  $V$  be an  $I$ -graded free  $R$ -module with  $\dim V = \alpha$ . Define the scheme over  $R$

$$\mathcal{R}(\Gamma, \alpha) := \bigoplus_{h \in \Gamma} \text{Hom}(V_{h'}, V_{h''}).$$

Then we can identify  $\mathcal{R}(\bar{\Gamma}, \alpha)$  with  $\mathcal{R}(\Gamma, \alpha) \oplus \mathcal{R}(\Gamma, \alpha)^*$ . There is an obvious action of the group

$$\text{GL}_\alpha := \prod_{i \in I} \text{GL}_{\alpha_i}$$

on  $\mathcal{R}(\Gamma, \alpha)$  and therefore on  $\mathcal{R}(\bar{\Gamma}, \alpha)$  (this action can be factored through  $G_\alpha = \text{GL}_\alpha / \mathbb{G}_m$ , where  $\mathbb{G}_m$  is considered as a diagonal subgroup in  $\text{GL}_\alpha$ ). There is a map (which is a moment map if  $R$  is a field)

$$\mu : \mathcal{R}(\bar{\Gamma}, \alpha) \rightarrow \mathfrak{g}_\alpha^* \hookrightarrow \mathfrak{gl}_\alpha^*$$

defined by

$$(x_h)_{h \in \bar{\Gamma}} \mapsto \sum_{h \in \Gamma} [x_h, x_{\bar{h}}],$$

where  $\mathfrak{gl}_\alpha = \prod_i M_{\alpha_i \times \alpha_i}$  is a Lie algebra of  $\text{GL}_\alpha$  and  $\mathfrak{gl}_\alpha^*$  is isomorphic to  $\mathfrak{gl}_\alpha$  by the trace pairing;  $\mathfrak{g}_\alpha$  is a Lie algebra of  $G_\alpha$  and  $\mathfrak{g}_\alpha^* \hookrightarrow \mathfrak{gl}_\alpha^*$  can be identified with such matrices  $(\xi_i)_{i \in I}$  that  $\sum_i \text{tr } \xi_i = 0$ .

**Definition 2.1.** We call an element  $x \in \mathcal{R}(\Gamma, \alpha)$  nilpotent, if there exists some  $N \geq 1$  such that for any path  $h_1 \dots h_N$  in  $\Gamma$  (i.e.  $h_i'' = h_{i+1}'$ ,  $1 \leq i < N$ ) it holds  $x_{h_N} \dots x_{h_1} = 0$ .

Now we recall some facts from the paper of King [11] about the moduli spaces of semistable representations of quivers. In his paper King uses the geometric invariant theory over an algebraically closed field. According to Seshadri [19] this can be done also over  $\mathbb{Z}$ . The quotients obtained by Seshadri are categorical quotients but in general not universal categorical quotients. This problem was overcome in [4, Lemma B.4], where it was shown that after base change from  $\mathbb{Z}$  to some  $\mathbb{Z}_f$ ,  $f \in \mathbb{Z}$  the quotients are universal categorical quotients. This implies that we can consider these quotients over  $\mathbb{C}$  and over finite fields of a sufficiently large characteristic. In what follows, we will avoid all this formalities and describe the constructions over an algebraically closed field, but we will bear in mind that everything can be done over  $\mathbb{Z}_f$ . This will be used later.

Let  $\mathbb{F}$  be an algebraically closed field. For any  $\theta \in \mathbb{Z}^I$  with  $\theta \cdot \alpha = 0$ , we define a character  $\chi_\theta : \text{GL}_\alpha \rightarrow \mathbb{G}_m$  by the formula  $\chi_\theta(g) = \prod_{i \in I} \det(g_i)$ . This character defines an action of  $\text{GL}_\alpha$  on the trivial line bundle  $L$  over  $\mathcal{R}(\Gamma, \alpha)$ .

**Definition 2.2.** A point  $x \in \mathcal{R}(\Gamma, \alpha)$  is called  $\theta$ -stable (respectively,  $\theta$ -semistable) if for any  $I$ -graded,  $x$ -invariant subspace  $0 \neq V' \subsetneq V$  it holds  $\theta \cdot \dim V' > 0$  (respectively,  $\theta \cdot \dim V' \geq 0$ ).

It is proved in [11] that the stable part  $\mathcal{R}(\Gamma, \alpha)^s$  (respectively, semistable part  $\mathcal{R}(\Gamma, \alpha)^{ss}$ ) of  $\mathcal{R}(\Gamma, \alpha)$  with respect to the  $\text{GL}_\alpha$ -line bundle  $L$  consists precisely of

$\theta$ -stable (respectively  $\theta$ -semistable points). By the geometric invariant theory (see [11] for details) there exists a geometric quotient  $\mathcal{R}(\Gamma, \alpha)^s / G_\alpha$  and a categorical quotient  $\mathcal{R}(\Gamma, \alpha)^{ss} // G_\alpha$ . Moreover, the inclusion  $\mathcal{R}(\Gamma, \alpha)^{ss} \hookrightarrow \mathcal{R}(\Gamma, \alpha)$  induces a projective map  $\mathcal{R}(\Gamma, \alpha)^{ss} // G_\alpha \rightarrow \mathcal{R}(\Gamma, \alpha) // G_\alpha$ . In the same way one defines stability conditions and the corresponding moduli spaces for the representations of  $\bar{\Gamma}$ .

**Lemma 2.3.** *The moment map  $\mu : \mathcal{R}(\bar{\Gamma}, \alpha)^s \rightarrow \mathfrak{g}_\alpha^*$  is smooth.*

*Proof.* Stability condition implies that the stabilizer in  $G_\alpha$  of any stable point is trivial. This implies that  $\mu$  is smooth at any stable point.  $\square$

**Corollary 2.4.** *If  $\mu^{-1}(0)^s$  is nonempty then the map  $\mu : \mathcal{R}(\bar{\Gamma}, \alpha)^s \rightarrow \mathfrak{g}_\alpha^*$  is surjective.*

*Proof.* It is known that the smooth morphisms are open, so the image of the map  $\mu : \mathcal{R}(\bar{\Gamma}, \alpha)^s \rightarrow \mathfrak{g}_\alpha^*$  is open. As this image contains 0 and is stable with respect to the multiplication by scalars, it coincides with  $\mathfrak{g}_\alpha^*$ .  $\square$

**Lemma 2.5.** *Assume that  $\mathcal{R}(\bar{\Gamma}, \alpha)^s = \mathcal{R}(\bar{\Gamma}, \alpha)^{ss}$  and there exists some  $G_\alpha$ -invariant element  $\xi \in \mathfrak{g}_\alpha^*$  with  $\mu^{-1}(\xi)^s \neq \emptyset$ . Then the map  $\mu : \mathcal{R}(\bar{\Gamma}, \alpha)^s \rightarrow \mathfrak{g}_\alpha^*$  is surjective and for finite fields  $\mathbb{F}$  of a sufficiently large characteristic we have  $\#\mu^{-1}(\xi)^s // G_\alpha(\mathbb{F}) = \#\mu^{-1}(0)^s // G_\alpha(\mathbb{F})$ .*

*Proof.* We have to prove just the second statement. It is a consequence of [4, Appendix by Nakajima]. Let  $L$  be a line through 0 and  $\xi$  in  $\mathfrak{g}_\alpha^*$ ,  $X := \mu^{-1}(L)^s$  and  $Y = \mu^{-1}(L)$ . Let  $\mathcal{X} = X // G_\alpha$  and  $\mathcal{Y} := Y // G_\alpha$ . There is a commutative diagram

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\pi} & \mathcal{Y} \longrightarrow L, \end{array}$$

where  $\pi$  is projective. Let  $\mathbb{G}_m$  acts on  $\mathcal{R}(\Gamma, \alpha)$  by multiplication of all the matrices by scalar. This action induces the action of  $\mathbb{G}_m$  on  $X$ ,  $Y$ ,  $\mathcal{X}$ ,  $\mathcal{Y}$ . There is also an action of  $\mathbb{G}_m$  on  $L$  s.t. the maps in the above diagram are  $\mathbb{G}_m$ -equivariant. We claim that for any  $x \in \mathcal{X}$  there exists  $\lim_{t \rightarrow 0} tx$ . First of all, the map  $\mathbb{G}_m \rightarrow \mathcal{Y}$ ,  $t \mapsto t\pi(x)$  can be extended to  $\mathbb{A}^1$  by  $0 \mapsto 0$ . Now it follows from the projectivity of  $\pi$  that the map  $\mathbb{G}_m \rightarrow \mathcal{X}$ ,  $t \mapsto tx$  can also be extended to  $\mathbb{A}^1 \rightarrow \mathcal{X}$ . This proves the existence of the limit. We apply now the result of [4, Appendix by Nakajima] to the smooth family  $\mathcal{X} \rightarrow L$  with the above  $\mathbb{G}_m$ -action and get the second statement.  $\square$

### 3. QUIVER VARIETIES

We assume that  $\mathbb{F}$  is an algebraically closed field. Let  $\nu \in \mathbb{N}^I$  be fixed and let  $W$  be an  $I$ -graded vector space of dimension  $\nu$ . Let  $\Gamma_*$  be the quiver defined in Introduction; that is, we adjoin to  $\Gamma$  a new vertex  $*$  and  $\nu_i$  arrows from  $*$  to  $i$  for each  $i \in I$ . We identify  $W_i$  with  $\bigoplus_{h: * \rightarrow i} \mathbb{F} \cdot h$ . Let as before  $V$  be an  $I$ -graded vector space of dimension  $\alpha$ . As in Introduction, a pair  $(\alpha, n)$  with  $\alpha \in \mathbb{N}^I$  and  $n \in \mathbb{N}$  will be considered as an element from  $\mathbb{N}^{I*}$ .

There is an obvious identification

$$M(\alpha, \nu) := \mathcal{R}(\bar{\Gamma}_*, (\alpha, 1)) = \mathcal{R}(\bar{\Gamma}, \alpha) \oplus \text{Hom}_I(W, V) \oplus \text{Hom}_I(V, W).$$

The elements of this space will be represented as triples  $(x, p, q)$ . Note that  $G_{(\alpha, 1)} = (\prod_{i \in I} \mathrm{GL}_{\alpha_i} \times \mathbb{G}_m) / \mathbb{G}_m \simeq \mathrm{GL}_{\alpha}$ . Therefore the moment map may be considered as a map  $\mu_* : M(\alpha, \nu) \rightarrow \mathfrak{gl}_{\alpha}^*$ . It is given by the formula

$$\mu_*(x, p, q) = \mu(x) + pq.$$

We fix once and for all  $\theta \in \mathbb{Z}^{I_*}$ ,  $\theta = (-1, \dots, -1, \sum \alpha_i)$  and consider stability and semistability conditions in  $M(\alpha, \nu)$  with respect to  $\theta$ .

**Lemma 3.1.** *Stability and semistability conditions in  $M(\alpha, \nu)$  are equivalent. An element  $(x, p, q) \in M(\alpha, \nu)$  is stable if and only if any  $I$ -graded,  $x$ -invariant subspace  $V' \subset V$  s.t.  $q(V') = 0$  is zero.*

*Proof.* Assume that  $(x, p, q)$  is semistable and there exists an  $I$ -graded,  $x$ -invariant subspace  $V' \subset V$  s.t.  $q(V') = 0$ . Then  $V'$  may be considered as an  $I_*$ -graded,  $(x, p, q)$ -invariant subspace of  $V \oplus \mathbb{F}$  of dimension  $(\dim V', 0)$ . From semistability condition we get  $-\dim V' \geq 0$ , thus  $V' = 0$ . It follows that the last condition of the lemma holds.

Conversely, assume that the last condition of the lemma holds. Let  $V' \oplus V_*$  be some proper,  $I_*$ -graded,  $(x, p, q)$ -invariant subspace of  $V \oplus \mathbb{F}$ . If  $V_* = \mathbb{F}$ , then  $\dim V' < \alpha$  and therefore  $\theta \cdot (\dim V', 1) = \sum_i \alpha_i - \sum_i \dim V'_i > 0$ . If  $V_* = 0$ , then  $q(V') = 0$  and by our assumption  $V' = 0$ . This implies that  $(x, p, q)$  is stable.  $\square$

**Remark 3.2.** For any subscheme  $X \subset M(\alpha, \nu)$  we write respectively  $X^n$ ,  $X^s$ ,  $X^{ns}$  to denote the subschemes of  $X$  consisting respectively of nilpotent, stable, nilpotent and stable elements.

**Definition 3.3.** We define the quiver variety  $\mathcal{M} = \mathcal{M}(\alpha, \nu)$  to be the quotient  $\mu_*^{-1}(0)^s // \mathrm{GL}_{\alpha}$ . Define  $\mathcal{L} = \mathcal{L}(\alpha, \nu) := \mu_*^{-1}(0)^{ns} // \mathrm{GL}_{\alpha}$ .

**Remark 3.4.** It is easy to see that  $\mathcal{L}(\alpha, \nu)$  is the preimage of zero under the projective morphism  $\mu_*^{-1}(0)^s // \mathrm{GL}_{\alpha} \rightarrow \mu_*^{-1}(0) // \mathrm{GL}_{\alpha}$ . It is known that an element  $(x, p, q) \in M(\alpha, \nu)^s$  is nilpotent if and only if  $x$  is nilpotent and  $p = 0$ , see e.g. [15, Lemma 5.9] or [12, Lemma 2.22].

Let  $T$  denotes the Tits form of the quiver  $\Gamma$  and  $T_*$  denotes the Tits form of the quiver  $\Gamma_*$ . As in Introduction, we define  $d = d(\alpha, \nu) := 1 - T_*(\alpha, 1) = \alpha \cdot \nu - T(\alpha)$ .

**Theorem 3.5** (Nakajima [16, Section 3]). *Variety  $\mathcal{M}$  is smooth and variety  $\mathcal{L}$  is projective. The complex manifold  $\mathcal{M}(\mathbb{C})$  is symplectic and its subvariety  $\mathcal{L}(\mathbb{C})$  is a Lagrangian subvariety homotopic to  $\mathcal{M}(\mathbb{C})$ . The dimension of  $\mathcal{M}$  equals  $2d(\alpha, \nu)$ .*

For a scheme  $X$  of finite type over  $\mathbb{Z}_f$ , we define

$$h^i(X) := \dim H^i(X(\mathbb{C}), \mathbb{Q}), \quad h_c^i(X) := \dim H_c^i(X(\mathbb{C}), \mathbb{Q}).$$

Note that  $h_c^i(X)$  can also be defined as  $\dim H_c^i(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$  (for large enough prime  $p$ ) by the base change theorem [5, Theorem 1.8.7] and comparison theorem [5, Theorem 1.11.6].

**Lemma 3.6.**  $\dim L(\overline{\nu})_{\overline{\nu}-\alpha} = h_c^{2d}(\mathcal{M})$ .

*Proof.* It is well-known (see e.g. [16] or [18]) that  $\dim L(\overline{\nu})_{\overline{\nu}-\alpha}$  equals the number of irreducible components of  $\mathcal{L}$  i.e.,  $h_c^{2d}(\mathcal{L})$ . We note that

$$h_c^{2d}(\mathcal{L}) = h^{2d}(\mathcal{L}) = h^{2d}(\mathcal{M}) = h_c^{2d}(\mathcal{M}),$$

where the last equality follows from the Poincaré duality.  $\square$

## 4. MAIN RESULTS

Recall from Introduction that for any  $\alpha \in \mathbb{N}^I$  there is a polynomial  $a_\alpha(\Gamma) \in \mathbb{Z}[q]$  such that for any finite field  $\mathbb{F}_q$ ,  $a_\alpha(\Gamma, q)$  is the number of absolutely indecomposable representations of  $\Gamma$  over  $\mathbb{F}_q$  of dimension  $\alpha$ . As before  $\mathcal{M} = \mathcal{M}(\alpha, \nu)$  is a quiver variety.

**Proposition 4.1.** *For a finite field  $\mathbb{F}_q$  of a sufficiently large characteristic it holds  $\#\mathcal{M}(\mathbb{F}_q) = q^{d(\alpha, \nu)} a_{(\alpha, 1)}(\Gamma_*, q)$ .*

*Proof.* We are going to apply [4, Proposition 2.2.1]. Let  $\alpha' := (\alpha, 1) \in \mathbb{N}^{I*}$ . Note that  $\theta = (-1, \dots, -1, \sum \alpha_i)$  is  $\alpha'$ -generic in the sense that for any  $0 < \gamma' < \alpha'$  it holds  $\theta \cdot \gamma' \neq 0$ . Indeed, let  $\gamma' = (\gamma, k)$ , where  $\gamma \in \mathbb{Z}^I$  and  $k \in \{0, 1\}$ . If  $k = 0$  then clearly  $\theta \cdot \gamma' = -\sum_i \gamma_i < 0$ . If  $k = 1$  then  $\gamma < \alpha$  and  $\theta \cdot \gamma' = \sum_i \alpha_i - \sum_i \gamma_i > 0$ . We identify  $\theta$  with an element of  $\mathfrak{gl}_{\alpha'}$ , consisting of diagonal matrices. Then all the points of  $\mu_*^{-1}(\theta)$  are stable. Indeed, assume that some  $(x, p, q)$  is not stable i.e., there exists an  $I$ -graded,  $x$ -invariant subspace  $0 \neq V' \subsetneq V$  with  $q(V') = 0$ . Then it follows that  $\sum_{h \in \Gamma} [x_h, x_{\bar{h}}]|_{V'} = (-\text{Id}_{V'_i})_{i \in I}$ . Adding the traces we get  $0 = \sum_i \dim V'_i$ , which is impossible. Applying now [4, Proposition 2.2.1], we get  $a_{\alpha'}(\Gamma_*, q) = q^{-d(\alpha, \nu)} \cdot \#\mu_*^{-1}(\theta) // \text{GL}_\alpha(\mathbb{F}_q)$  for a finite field  $\mathbb{F}_q$  of a sufficiently large characteristic. By Lemma 2.5 we have  $\#\mu_*^{-1}(\theta) // \text{GL}_\alpha(\mathbb{F}_q) = \#\mathcal{M}(\mathbb{F}_q)$ .  $\square$

It follows that there exists a polynomial  $p(\mathcal{M}) \in \mathbb{Z}[q]$  such that for finite fields  $\mathbb{F}_q$  of large enough characteristic it holds  $\#\mathcal{M}(\mathbb{F}_q) = p(\mathcal{M}, q)$ .

*Proof of Theorem 2.* According to [4, Lemma A.1] and [4, Proposition A.2] it holds  $p(\mathcal{M}, q) = \sum_i h_c^{2i}(\mathcal{M}) q^i$  and  $h_c^{2i+1}(\mathcal{M}) = 0$  whenever we show the existence of a  $\mathbb{G}_m$ -action on  $\mathcal{M}$  s.t. for any  $x \in \mathcal{M}$  there exists  $\lim_{t \rightarrow 0} tx$  and s.t.  $\mathcal{M}^{\mathbb{G}_m}$  is projective. This action was described in Lemma 2.5. Note that the  $\mathbb{G}_m$ -invariant part of  $\mathcal{M}$  is mapped to zero under the map  $\pi : \mathcal{M} \rightarrow \mu_*^{-1}(0) // \text{GL}_\alpha$ . This implies that  $\mathcal{M}^{\mathbb{G}_m} \subset \mathcal{L}$  and therefore  $\mathcal{M}^{\mathbb{G}_m}$  is projective.  $\square$

*Proof of Theorem 1.* By the Kac conjecture, proved for indivisible vectors in [4], it holds  $\dim \mathfrak{g}(\Gamma_*)_{(\alpha, 1)} = a_{(\alpha, 1)}(\Gamma_*, 0)$ . By Proposition 4.1 it holds  $a_{(\alpha, 1)}(\Gamma_*, 0) = q^{-d(\alpha, \nu)} p(\mathcal{M}, q)|_{q=0}$ . So, we have to prove  $\dim L(\bar{\nu})_{\bar{\nu}-\alpha} = q^{-d(\alpha, \nu)} p(\mathcal{M}, q)|_{q=0}$ . From the facts that  $\mathcal{M}$  is homotopic to  $\mathcal{L}$  and that  $\mathcal{L}$  is projective of dimension  $d(\alpha, \nu)$  we get that  $h^i(\mathcal{M}) = h^i(\mathcal{L}) = h_c^i(\mathcal{L}) = 0$  for  $i > 2d(\alpha, \nu)$  and therefore  $h_c^i(\mathcal{M}) = 0$  for  $i < 2d(\alpha, \nu)$  by Poincaré duality. It follows that  $h_c^{2d(\alpha, \nu)}(\mathcal{M}) = q^{-d(\alpha, \nu)} p(\mathcal{M}, q)|_{q=0}$  and we apply Lemma 3.6.  $\square$

## 5. POINCARÉ POLYNOMIALS OF QUIVER VARIETIES

In this section we recall the explicit formula for the functions  $r_\alpha(\Gamma, q)$  defined in Introduction and the relation between  $r(\Gamma, q)$  and  $a(\Gamma, q)$ , see [8, 14]. From this relation and Theorem 2 we derive then Theorem 3.

Let  $(\Gamma, I)$  be a finite quiver. Let  $\mathcal{P}$  be the set of partitions (see e.g. [13]) and let  $\lambda = (\lambda^i)_{i \in I} \in \mathcal{P}^I$ . Define  $|\lambda| := (|\lambda^i|)_{i \in I} \in \mathbb{N}^I$ . For any  $j \geq 1$  define  $\lambda_j := (\lambda_j^i)_{i \in I} \in \mathbb{N}^I$ . Define  $T(\lambda) := \sum_{j \geq 1} T(\lambda_j)$ , where the quadratic form  $T$  on  $\mathbb{Z}^I$  is the Tits form defined in Introduction. Then the function  $r_\alpha(\Gamma) \in \mathbb{Q}(q)$ ,  $\alpha \in \mathbb{N}^I$  defined

in Introduction equals

$$r_\alpha(\Gamma, q) := \sum_{|\lambda|=\alpha} \frac{q^{-T(\lambda)}}{\prod_{i \in I} \varphi_{\lambda^i}(q^{-1})},$$

where  $\varphi_\mu(q) := \prod_{j \geq 1} \varphi_{\mu_j - \mu_{j+1}}(q)$  for  $\mu \in \mathcal{P}$  and  $\varphi_n(q) := (1 - q) \dots (1 - q^n)$  for  $n \in \mathbb{N}$ .

To describe the relation between  $a(\Gamma) = \sum_\alpha a_\alpha(\Gamma)x^\alpha$  and  $r(\Gamma) = \sum_\alpha r_\alpha(\Gamma)x^\alpha$ , we use  $\lambda$ -rings (see e.g. [14, Appendix]). We endow the field  $\mathbb{Q}(q)$  with the structure of a  $\lambda$ -ring in terms of Adams operations by

$$\psi_n(f(q)) := f(q^n), \quad f \in \mathbb{Q}(q).$$

In order to avoid any problems with the Adams operations in what follows, we tensor our  $\lambda$ -rings with  $\mathbb{Q}$  without mentioning that and so we always assume that our  $\lambda$ -rings contain  $\mathbb{Q}$ . If  $R$  is a  $\lambda$ -ring, we endow the ring  $R[x_1, \dots, x_r]$  with a  $\lambda$ -ring structure by the formula  $\psi_n(ax^\alpha) := \psi_n(a)x^{n\alpha}$ , where  $a \in R$ ,  $\alpha \in \mathbb{N}^r$ . In the same way, we endow with a  $\lambda$ -ring structure the ring of formal power series over  $R$ . Given a  $\lambda$ -ring  $R$ , we define the map  $\text{Exp} : R[[x_1, \dots, x_r]]^+ \rightarrow 1 + R[[x_1, \dots, x_r]]^+$  (here  $R[[x_1, \dots, x_r]]^+$  is an ideal  $(x_1, \dots, x_r)$ ) by the formula  $\text{Exp}(f) := \sum_{n \geq 0} \sigma_n(f)$  or, in terms of Adams operations,

$$\text{Exp}(f) = \exp\left(\sum_{k \geq 1} \frac{1}{k} \psi_k(f)\right),$$

where the map  $\exp$  (as well as the map  $\log$  used below) is defined as in [1, Ch.II §6]. The map  $\text{Exp}$  has an inverse  $\text{Log} : 1 + R[[x_1, \dots, x_r]]^+ \rightarrow R[[x_1, \dots, x_r]]^+$  which is given by the formula of Cadogan (see [2, 6, 14])

$$\text{Log}(f) = \sum_{k \geq 1} \frac{\mu(k)}{k} \psi_k(\log(f)),$$

where  $\mu$  is a Möbius function. Now we are ready to write down the relation between  $a(\Gamma, q)$  and  $r(\Gamma, q)$  (see [14, Theorem 2])

$$a(\Gamma, q) = (q - 1) \text{Log}(r(\Gamma, q)).$$

This formula together with Theorem 2 enables us to calculate the Poincaré polynomial of quiver varieties. The content of Theorem 3 is a formula for the direct computation of the Poincaré polynomial of quiver varieties.

*Proof of Theorem 3.* For any  $n \in \mathbb{N}$ , we define

$$a_n := \sum_{\alpha \in \mathbb{N}^I} a_{(\alpha, n)}(\Gamma_*)x^\alpha, \quad r_n := \sum_{\alpha \in \mathbb{N}^I} r_{(\alpha, n)}(\Gamma_*)x^\alpha,$$

$a_* := \sum_{n \geq 0} a_n x_*^n$  and  $r_* := \sum_{n \geq 0} r_n x_*^n$ . Then by Theorem 2 it holds

$$\sum_{\alpha} q^{-d(\alpha, \nu)} p(\mathcal{M}(\alpha, \nu), q) x^\alpha = a_1(q).$$

We know that  $a_*(q) = (q-1)\text{Log}(r_*(q))$  and therefore

$$\begin{aligned} a_1(q) &= \frac{\partial}{\partial x_*} a_*(q) \Big|_{x_*=0} = (q-1) \frac{\partial}{\partial x_*} \sum_{k \geq 1} \frac{\mu(k)}{k} \psi_k(\log(r_*(q))) \Big|_{x_*=0} \\ &= (q-1) \frac{\partial}{\partial x_*} \log(r_*(q)) \Big|_{x_*=0} = (q-1) \frac{\frac{\partial}{\partial x_*} r_*(q)}{r_*(q)} \Big|_{x_*=0} = (q-1) \frac{r_1(q)}{r_0(q)}. \end{aligned}$$

We note that  $r_0(q) = r(\Gamma, q)$ . □

#### REFERENCES

- [1] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 1–3*, Elements of Mathematics, Springer-Verlag, Berlin, 1989.
- [2] Charles Cadogan, *The Möbius function and connected graphs*, J. Comb. Theory, Ser. B **11** (1971), 193–200.
- [3] William Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compos. Math. **126** (2001), no. 3, 257–293.
- [4] William Crawley-Boevey and Michel Van den Bergh, *Absolutely indecomposable representations and Kac-Moody Lie algebras*, Invent. Math. **155** (2004), no. 3, 537–559, With an appendix by Hiraku Nakajima.
- [5] Eberhard Freitag and Reinhardt Kiehl, *Etale cohomology and the Weil conjecture*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 13, Springer-Verlag, Berlin etc., 1988.
- [6] Ezra Getzler, *Mixed Hodge structures of configuration spaces*, Preprint 96-61, Max-Planck-Institut f. Mathematik, Bonn, arXiv:alg-geom/9510018.
- [7] Tamas Hausel, *Betti numbers of holomorphic symplectic quotients via arithmetic Fourier transform*, arXiv:math.AG/0511163.
- [8] Jiuzhao Hua, *Counting representations of quivers over finite fields*, J. Algebra **226** (2000), no. 2, 1011–1033.
- [9] Victor G. Kac, *Root systems, representations of quivers and invariant theory*, Invariant theory (Montecatini, 1982), Lecture Notes in Math., vol. 996, Springer, Berlin, 1983, pp. 74–108.
- [10] ———, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990.
- [11] A. D. King, *Moduli of representations of finite-dimensional algebras*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 180, 515–530.
- [12] G. Lusztig, *On quiver varieties*, Adv. Math. **136** (1998), no. 1, 141–182.
- [13] I. G. Macdonald, *Symmetric functions and Hall polynomials*, The Clarendon Press Oxford University Press, New York, 1979, Oxford Mathematical Monographs.
- [14] Sergey Mozgovoy, *A computational criterion for the Kac conjecture*, arXiv:math.RT/0608321.
- [15] Hiraku Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.
- [16] ———, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560.
- [17] Dale Peterson, *Freudenthal-type formulas for root and weight multiplicities*, Preprint.
- [18] Yoshihisa Saito, *Crystal bases and quiver varieties*, Math. Ann. **324** (2002), no. 4, 675–688.
- [19] C.S. Seshadri, *Geometric reductivity over arbitrary base*, Adv. Math. **26** (1977), 225–274.
- [20] Bert Sevenhant and Michel Van den Bergh, *On the number of absolutely indecomposable representations of a quiver*, J. Algebra **221** (1999), no. 1, 29–49.

INSTITUT FÜR MATHEMATIK, JOHANNES GUTENBERG-UNIVERSITÄT MAINZ, 55099 MAINZ, GERMANY.

*E-mail address:* mozgov@mathematik.uni-mainz.de